The Exact Interface Model for Wetting in the Two-Dimensional Ising Model[1]

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Abstract

We use *exact* methods to derive an interface model from an underlying microscopic model, i.e., the Ising model on a square lattice. At the wetting transition in the two-dimensional Ising model the long Peierls contour (or interface) gets depinned from the substrate. Using *exact* transfer-matrix methods, we find that on sufficiently large length scales (i.e., length scales sufficiently larger than the bulk correlation length) the distribution of the long contour is given by a *unique* probability measure corresponding to a continuous "interface model". The interface binding "potential" is a Dirac delta function with support on the substrate and therefore a *distribution* rather than a function. More precisely, critical wetting in the two-dimensional Ising model, viewed on length scales sufficiently larger than the bulk correlation length, is described by a *reflected* Brownian motion with a Dirac δ perturbation on the substrate so that exactly at the wetting transition the substrate is a perfectly *reflecting surface*, otherwise there exists a δ perturbation. A lattice solid-on-solid model was found to give identical results (albeit with modified parameters) on length scales sufficiently larger than the lattice spacing, thus demonstrating the universality of the continuous interface model.

KEY WORDS: critical wetting; exact results; interface models; Ising models; solid-on-solid models.

1 Introduction

The modern theory of wetting, viewed as a bona fide thermodynamical phase transition, was initiated by Cahn [1] who provided a mean-field description from Landau theory. This was later developed more extensively by Nakanishi and Fisher [2]. Abraham [3] analysed the wetting transition in a two-dimensional Ising model using exact methods and found behaviour close to the critical wetting temperature, T_{w}, very different from that predicted by mean-field theory. Although these studies included the "bulk" degrees of freedom, it quickly became apparent that significant progress in describing wetting in three-dimensional systems beyond mean-field theory was most likely to be achieved through the use of *interface models* [4] – [10].

The basic idea behind the interfacial description is to *coarse-grain* to sufficiently large length scales, such as the *bulk* correlation length, ξ_{b}, so that the only fluctuating degrees of freedom left are the heights of the wetting interface, $y(\mathbf{x}) \geq 0$, above points $\mathbf{x}$ in the substrate $S \subset \mathbf{R}^{d-1}$ (d is the bulk dimension). One then arrives at an effective Hamiltonian, $\mathcal{H}_{\mathrm{eff}}[y]$, usually given as

$$\mathcal{H}_{\mathrm{eff}}[y] = \int_S d\mathbf{x} \left[\frac{1}{2}\tilde{\tau}|\boldsymbol{\nabla} y|^2 + V(y)\right] \tag{1}$$

where $\tilde{\tau}$ is the interfacial stiffness. Throughtout this paper we shall only be considering systems with short-ranged forces and for these the interfacial potential, $V(y)$, was originally given the form [4, 5],

$$V(y) = v_1(T)\, e^{-y/\xi_{\mathrm{b}}} + v_2\, e^{-2y/\xi_{\mathrm{b}}} + \ldots \tag{2}$$

where v_2 is positive and usually taken to be independent of temperature T and $v_1(T) \propto T - T_{\mathrm{w}}^{\mathrm{mf}}$ with $T_{\mathrm{w}}^{\mathrm{mf}}$ being the critical wetting temperature as determined by mean-field theory. More systematic approaches starting from an underlying Landau-Ginzburg-Wilson Hamiltonian followed [8, 9] which led to $V(y)$ given by (2) but with prefactors polynomial in y preceding the exponentials. These studies also opened up the possibility of a y-dependent stiffness $\tilde{\tau}$. In any case, the partitition function, Z_S, is given by the functional integral

$$Z_S = \prod_{\mathbf{x}\in S} \int_0^\infty dy(\mathbf{x})\, e^{-\mathcal{H}_{\rm eff}[y]} \tag{3}$$

but it should be stressed that this is only a formal expression, whose precise mathematical meaning is unclear, and contained within it is some lower-length cut-off. A description of critical wetting which goes beyond mean-field theory is then obtained by applying "functional renormalization group" methods [6, 7, 8, 10] — mean-field theory follows from minimizing $\mathcal{H}_{\rm eff}[y]$. The following questions concerning interface models come to mind.

(1) Does any of this make sense mathematically and can such interface models be *derived* using *exact* methods? Previous derivations, although careful, are somewhat heuristic and essentially mean-field in character. A more rigorous approach would be desirable.

(2) What length scales are they valid for? In order to "smear out" bulk fluctuations one would have thought it necessary to coarse grain to a scale of *at least* the bulk correlation length, $\xi_{\rm b}$, which would then serve as a lower-length "cut-off" to the functional integrals. At the same time, $\xi_{\rm b}$ appears explicitly in the expression for $V(y)$ as given by Eq. (2) and thus determines the range at which $V(y)$ acts — a range that is *no bigger* than the cut-off scale.

(3) How much information is contained in these models? For instance, can interface models determine the critical properties of correlation functions as well as thermodynamic singularites for critical wetting?

In this paper, we attempt to answer these questions on a more rigorous footing through an exact analysis of a two-dimensional Ising model. For comparison, we also give analogous results for a *lattice* solid-on-solid (SOS) model (also in bulk two dimensions). Roughly speaking, our main result is that, provided one coarse grains to length scales *sufficiently* larger than ξ_{b}, a "continuous" interface model similar to the above does indeed describe critical wetting except that in our case we find that the "interface potential" is given by $V(y) = c\,\delta_0(y)$ where $\delta_0(\cdot)$ is the Dirac delta *distribution* supported on $\{0\}$ and c depends on temperature and the various microscopic parameters of the underlying model. Furthermore, $c > 0$ (i.e., repulsive substrate) when $T > T_{\mathrm{w}}$; $c < 0$ (attractive substrate) when $T < T_{\mathrm{w}}$; and $c = 0$ when $T = T_{\mathrm{w}}$.

In Section 2 we describe the microscopic models considered and in Section 3 present the resulting interface model that these limit to in a more precise form. A brief outline of the methodology used to get this result is laid out in Section 4. Further details of some of this analysis are given elsewhere [11]. Finally, we finish with some conclusions in Section 5.

2 Microscopic Models

2.1 Two-Dimensional Ising Model

Ising spins, $\sigma_{m,n} = \pm 1$, are placed on sites (m, n) $(1 \le m \le M,\ 0 \le n \le N)$ of a square lattice $\Lambda \subset \mathbf{Z}^2$ wrapped on a cylinder of height $N + 1$ and circumference M (i.e., periodic boundary conditions in the m direction). The top of the cylinder $(n = N)$ may be left free. Following Abraham [3], two types of boundary conditions are imposed at the bottom of the cylinder $(n = 0)$. In Case $\mathcal{A}$ one fixes $\sigma_{m,0} = +1$ for *all* $1 \le m \le M$; for Case $\mathcal{B}$, $\sigma_{m,0} = -1$ for $1 \le m \le x$ and $\sigma_{m,0} = +1$ for $x + 1 \le m \le M$. The spins interact ferromagnetically across nearest neighbours according to the following Hamiltonian

$$\mathcal{H}_\Lambda(\sigma)/k_{\mathrm{B}}T = -\sum_{m=1}^{M}\left(K_1\sum_{n=1}^{N-1}\sigma_{m,n}\sigma_{m,n+1} + K_2\sum_{n=1}^{N}\sigma_{m,n}\sigma_{m+1,n} + h_1\sigma_{m,0}\sigma_{m,1}\right). \quad (4)$$

Note that, since $\sigma_{m,0}$ is held fixed for all m, h_1 acts like a surface field on the row of spins at $n = 1$. The boundary condition $\mathcal{B}$ induces a long Peierls contour (i.e., the interface) joining $(\frac{1}{2}, \frac{1}{2})$ to $(x + \frac{1}{2}, \frac{1}{2})$ on the dual lattice which is absent in Case $\mathcal{A}$. On defining

$$w := e^{2K_2}(\cosh 2K_1 - \cosh 2h_1)/ \sinh 2K_1, \tag{5}$$

Abraham [3] showed that, after taking the limits $M \to \infty$, $N \to \infty$ and $x \to \infty$ in that order, a wetting transition occurs in Case $\mathcal{B}$ at $w = 1$ with the interface being pinned (respectively de-pinned) when $w > 1$ (respectively $w < 1$). This wetting transition will show up thermodynamically as a singularity in the incremental free energy, $\tau^\times$, defined as

$$\tau^\times := - \lim_{x\to\infty} \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{x} \ln \left[Z^{\mathcal{B}}/Z^{\mathcal{A}} \right] \tag{6}$$

where $Z^{\mathbf{b}}$ is the canonical partition function for Case $\mathbf{b} = \mathcal{A}$, $\mathcal{B}$.

2.2 Solid-On-Solid Model

We consider the *lattice* model for an interface as introduced by Abraham and Smith [12]. The interfacial configurations consist of random "histograms" denoted by the Markov random field $Y = (Y_j)_{j=0}^x$ where $Y_j \in [0, \infty)$ is the height of the interface above the substrate at the lattice point j. The Gibbs measure for this system, $\mathbf{Q}_x(\cdot)$, is then given by

$$\mathbf{Q}_x(Y \in dy) = \frac{1}{Z_x} \exp\left(-\kappa \sum_{j=1}^{x} |y_j - y_{j-1}|\right) \left[\prod_{j=1}^{x-1} (1 + a\,\delta_0)(dy_j)\right] \delta_0(dy_0)\,\delta_0(dy_x) \tag{7}$$

where $\delta_0(dy_j) = \delta_0(y_j)dy_j$ denotes the Dirac measure at 0 and Z_x is the canonical partition function defined so that $\mathbf{Q}_x(\cdot)$ normalizes to 1. The tendency of the substrate to pin the interface increases with a. It was shown [12] that, in the limit $x \to \infty$, a wetting transition occurs at $a = 1/\kappa$ with the interface being pinned (repectively de-pinned) for $a > 1/\kappa$ (respectively $a < 1/\kappa$).

3 Main Results

The main purpose of this paper is determine from both lattice models of the previous section the probability distribution of the interface on length scales *sufficiently large* that the interface can be viewed as a continuous object, i.e., a path as a function on a continuous set. It is in this sense that we can talk about "continuous interface models". Thus, regarding the direction parallel to the substrate (x and s) as "time-like" and the height of interface above the substrate (y and Y_s) as "space-like", the Ising and SOS interface on a sufficiently large length scale will be treated as a *continuous-time* Markov stochastic process $(Y_s)_{s\in[0,x]}$ with $Y_0 = Y_x = 0$. In Subsection 3.1 the probability measure, $\mathbf{P}_x^c$, for $(Y_s)_{s\in[0,x]}$ will be presented and this will provide a more mathematically precise description of the emergent continuous interface model but first we need to define some of the quantities which enter $\mathbf{P}_x^c$.

Consider the tied-down Brownian motion $(B_s)_{s\in[x_1,x_2]}$ on $\mathbf{R}$ with $B_{x_1} = y_1$, $B_{x_2} = y_2$ [13] (this is sometimes called a Brownian bridge). Let the Brownian motion have diffusion constant $1/2\tilde{\tau}$. Its conditional probability measure, $\nu_{(x_1,y_1)}^{(x_2,y_2)}$, is the extension of the finite-dimensional distributions on $\mathbf{R}^n$ given as

$$\begin{aligned}
&\nu_{(x_1,y_1)}^{(x_2,y_2)}(\mathbf{R}^{[x_1,x_2]}|B_{s_1}\in db_1,\ldots,B_{s_n}\in db_n) \\
&\quad = \frac{g(s_1-x_1;b_1-y_1)g(x_2-s_n;y_2-b_n)\,db_1}{g(x_2-x_1;y_2-y_1)}\prod_{j=2}^{n} g(s_j-s_{j-1};b_j-b_{j-1})\,db_j
\end{aligned} \tag{8}$$

where $x_1 < s_1 < \ldots < s_n < x_2$ and $g(x;y)$ is the Gauss kernel

$$g(x;y) = \left(\frac{\tilde{\tau}}{2\pi x}\right)^{1/2} e^{-\tilde{\tau}y^2/2x}. \tag{9}$$

Now, tied-down *reflected* Brownian motion (reflected off $y=0$) is defined by the process $(|B_s|)_{s\in[x_1,x_2]}$ [13] which is assigned a conditional measure $\mu_{(x_1,y_1)}^{(x_2,y_2)}$ with the normalization $\int d\mu_{(x_1,y_1)}^{(x_2,y_2)} = g_- + g_+$ with $g_\pm = g(x_2-x_1;y_2\pm y_1)$. The probability measure $\mathbf{P}_x^c$ for $(Y_s)_{s\in[0,x]}$ will be defined in terms of $\mu_{(0,0)}^{(x,0)}$.

3.1 Exact Continuous Interface Model

Recall that the height of wetting interface on a large length scale is represented by the stochastic process $(Y_s \in [0,\infty))_{s\in[0,x]}$. It can be shown that, for both the Ising and SOS models, its probability measure $\mathbf{P}_x^c$ on the infinite-dimensional space $\Omega_x = [0,\infty)^{[0,x]}$ is given by

$$\mathbf{P}_x^c(\cdot) = \frac{1}{Z_x(c)} e^{-2cL_x} \mu_{(0,0)}^{(x,0)}(\cdot) \tag{10}$$

where the partition function, $Z_x(c)$, is the following path integral

$$Z_x(c) = \int d\mu_{(0,0)}^{(x,0)} e^{-2cL_x}. \tag{11}$$

The random variable L_x is the Brownian "local time" [13] defined by

$$L_x := \lim_{\epsilon \downarrow 0} \frac{1}{4\epsilon} \operatorname{meas}\{0 \le s \le x : Y_s \le \epsilon\} \tag{12}$$

where $\operatorname{meas}\{\cdot\}$ denotes the Lebesgue measure. Thus, L_x provides a measure of the amount of interface staying close to the substrate and *formally* it can be expressed in terms of the δ distribution as

$$2L_x = \int_0^x \delta_0(Y_s)\, ds. \tag{13}$$

The incremental free energy (which for the Ising model is defined by (6)) is now given by

$$\tau^\times(c) - \tau = -\lim_{x\to\infty} \frac{1}{x} \ln Z_x(c) \tag{14}$$

where τ is the interfacial tension for a free interface.

The measure $\mathbf{P}_x^c$ contains two parameters dependent on the underlying microscopic models; the interfacial stiffness $\tilde{\tau}$ (which incorporates lattice anisotropy [14, 15]) entering as the diffusion constant in $\mu_{(0.0)}^{(x,0)}$ and c. These are given as

$$\tilde{\tau} = \begin{cases} \sinh 2K_1^* \sinh 2K_2 \sinh \tau\,, & \text{Ising model;} \\ \frac{1}{2}\kappa^2\,, & \text{SOS model} \end{cases} \tag{15}$$

where the Ising interfacial tension τ is given by $\tau = 2(K_1 - K_2^*)$, $e^{-2K_j^*} = \tanh K_j$ and

$$c = \begin{cases} (1-w)/2\tilde{\tau}\,, & \text{Ising model;} \\ \frac{1}{\kappa} - a\,, & \text{SOS model} \end{cases} \tag{16}$$

recalling that w is given by (5).

The wetting transition occurs at $c = 0$ and the substrate is wet (respectively nonwet) when $c > 0$ (respectively $c < 0$). It should be stressed that the process $(Y_s)_{s\in[0,x]}$ with $\mathbf{P}_x^c$ provides an (asymptotically) *exact* description of the interface only on sufficiently large length scales. For the Ising model one requires that length scales be sufficiently larger than the bulk correlation length $\xi_{\mathrm{b}} = 1/2\tau$ and for the SOS model length scales need to be sufficiently larger than the SOS lattice spacing. Therefore, for $T < T_{\mathrm{w}}$, one requires that the wetting-layer thickness, ℓ, defined by the expectation $\ell = \lim_{x\to\infty} \mathbf{E}\, Y_{x/2}$, also be sufficiently large. So, for temperatures $T < T_{\mathrm{w}}$, this interfacial description is valid provided T be sufficiently close to T_{w}. For the Ising model this means that $w - 1$ must be sufficiently small when positive and similarly for $a - \frac{1}{\kappa}$ in the SOS model. However, for $T > T_{\mathrm{w}}$, the only restriction on T is that it be less than the bulk critical temperature T_{c} (which for the SOS model is effectively infinite).

Given Eq. (13), it is tempting to regard the interface model, quantum mechanically, as describing a Euclidean Schrödinger particle of mass $\tilde{\tau}$ moving on the half line $y \geq 0$ subject to a "potential" $c\delta_0(y)$. In doing so one needs to be clear on the effect of the boundary at $y = 0$ when $c = 0$. In other words, what is the underlying Markov process perturbed by the δ function? The answer is *reflected Brownian motion* since $\mathbf{P}_x^{c=0}$ is clearly the probability measure for tied-down reflected Brownian motion.

3.2 Family of Finite-Dimensional Distributions

It will prove useful to describe the family of finite-dimenional distributions which can be uniquely extended to the measure $\mathbf{P}_x^c$ (on the infinite-dimensional space Ω_x) presented in Subsection 3.1. Consider the cylinder set $\{A_j \subset [0,\infty)\}_{j=1}^n$ for all $n \geq 1$. Then the family of finite-dimensional distributions can be expressed as

$$\mathbf{P}_x^c(\Omega_x | Y_{x_1} \in A_1, \ldots, Y_{x_n} \in A_n) = \int_{A_1} dy_1 \ldots \int_{A_n} dy_n \, p_{x,n}(x_1, y_1; \ldots; x_n, y_n) \tag{17}$$

where $0 < x_1 < \cdots < x_n < x$ and $p_{x,n}(\cdot)$ is the joint probability density function given by

$$p_{x,n}(x_1, y_1; \ldots; x_n, y_n) = \frac{K(x_1; 0, y_1) K(x - x_n; y_n, 0)}{K(x; 0, 0)} \prod_{j=2}^{n} K(x_j - x_{j-1}; y_{j-1}, y_j) \tag{18}$$

with $K(\cdot)$ defined by the path integral

$$K(u; y_0, y) := \int d\mu_{(0,y_0)}^{(u,y)} \, e^{-2cL_u}. \tag{19}$$

By applying Dirichtet-form techniques [16, 17], the path integral $K(\cdot)$ can be shown to satisfy a Feynman-Kac formula in terms of the kernel of an evolution operator $e^{-u\hat{H}_c}$ through

$$K(u; y_0, y) = \left(\text{kernel}\, e^{-u\hat{H}_c}\right)(y_0, y) \tag{20}$$

where $\hat{H}_c$ is the operator on $L^2([0,\infty))$ given by

$$\hat{H}_c = \frac{-1}{2\tilde{\tau}} \Delta_{\mathrm{N}} + c\delta_0 \tag{21}$$

with δ_0 being the Dirac measure at 0 and Δ_{N} the one-dimensional Neumann Laplacian, $(\Delta_{\mathrm{N}}\psi)(y) = \psi''(y)$ with $\psi'(0) = 0$. The spectrum of $\hat{H}_c$ can be determined by treating the term $c\delta_0$ as a rank-1 perturbation on $-\Delta_{\mathrm{N}}/2\tilde{\tau}$ [18] and, hence, $K(\cdot)$ can be expressed in spectral form [19]

$$\begin{aligned} K(u; y_0, y) &= \Theta(-c) 4\tilde{\tau}|c| e^{2\tilde{\tau}c^2 u} e^{-2\tilde{\tau}|c|(y_0+y)} \\ &+ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-\omega^2 u/2\tilde{\tau}} \left[e^{i\omega(y_0-y)} - \left(\frac{2\tilde{\tau}c + i\omega}{2\tilde{\tau}c - i\omega}\right) e^{i\omega(y_0+y)} \right] \end{aligned} \tag{22}$$

where $\Theta(\cdot)$ is the Heaviside step function. Note that the first term in the RHS of (22) is due to the single bound state of $\hat{H}_c$ present whenever $c < 0$ but absent for all $c \geq 0$.

4 Brief Outline of the Analysis

The continuous interface model, specified by $\mathbf{P}_x^c$, was constructed from the underlying microscopic models by applying the *Kolmogorov extension theorem* [13] which in the present context states the following: the *consistent* family of finite-dimensional distributions given by the joint probability densities $\{p_{x,n}(\cdot)\}_{n\geq 1}$ implies the *existence* and *uniqueness* of the measure $\mathbf{P}_x^c$ on the infinite-dimensional space Ω_x. Therefore, our strategy is clear; for a given microscopic model one computes the joint probabilities for the interface (long contour) passing through *any* number ($n \geq 1$) of points on a sufficiently large scale. If these joint probabilities can be expressed in the form given by combining Eqs. (18) and (22) then this implies that the measure $\mathbf{P}_x^c$ *uniquely* provides the appropriate "continuum" description. We now sketch out how these joint probabilities were determined for each lattice model.

4.1 Ising Model

We start by considering joint probabilities of lattice-contour events evaluated in terms of the (horizontal) bond energy defined as $\mathcal{E}_{m,n} := \sigma_{m,n}\sigma_{m+1,n}$ so that $I_{m,n} := (1-\mathcal{E}_{m,n})/2$ is the indicator for a Peierls contour vertically crossing the bond joining (m,n) to $(m+1,n)$. Let $j \in X$ denote the lattice site (x_j, y_j) where X is the index set $X := \{1,\ldots,n\}$ (and n in X is *not* to be confused with the vertical lattice coordinate in (m,n)). Then, by introducing the notations $\mathcal{E}^X := \prod_{j\in X}\mathcal{E}_j$ and $I^X := \prod_{j\in X} I_j$, the *joint* probability of Peierls contours vertically crossing bonds at $\{(x_j+\frac{1}{2}, y_j)\}_{j\in X}$, with boundary condition $\mathbf{b} = \mathcal{A},\mathcal{B}$, is given by the canonical expectation $\langle I^X \rangle_{\mathbf{b}}$ (where, throughout, the limits $N, M \to \infty$ have already been taken). In the presence of a long contour, the probability $\langle I^X \rangle_{\mathcal{B}}$ can be shown to be given by

$$\langle I^X \rangle_{\mathcal{B}} = \sum_{X' \subseteq X} (-1/2)^{|X'|} \langle \mathcal{E}^{X'} \rangle_{\mathcal{B}}^{\text{con}} \langle I^{X \backslash X'} \rangle_{\mathcal{A}} \tag{23}$$

where $|X'|$ is the cardinality of the set X' and the sum includes the empty set, $\emptyset$, with the convention $I^{\emptyset} = \mathcal{E}^{\emptyset} = 1$. Also, $\langle \mathcal{E}^{X'} \rangle_{\mathcal{B}}^{\text{con}}$ is the *connected* and *rooted* $|X'|$-point bond-energy correlation function, truncated so that $\langle \mathcal{E}^{X'} \rangle_{\mathcal{B}}^{\text{con}} \to 0$ whenever $\max\{|x_j|, |x_j - x|\}_{j \in X'} \to \infty$. Unlike $\langle I^X \rangle_{\mathcal{B}}$, the joint probability $\langle I^X \rangle_{\mathcal{A}}$ is translationally invariant in the x direction, i.e., invariant under $\{x_j\}_{\forall j \in X} \mapsto \{x_j + u\}_{\forall j \in X}$, and can be written

$$\langle I^X \rangle_{\mathcal{A}} = \sum_{\varpi \in \mathcal{P}(X)} \prod_{P \in \varpi} \langle I^P \rangle_{\mathcal{A}}^T \quad \text{with} \quad \langle I^P \rangle_{\mathcal{A}}^T = (-1/2)^{|P|} \langle \mathcal{E}^P \rangle_{\mathcal{A}}^T \tag{24}$$

where $\mathcal{P}(X)$ is the set of all *partitions* of X, $\varpi = \{P_1, \ldots, P_{|\varpi|}\}$ is an element of $\mathcal{P}(X)$, P is an element of ϖ of $\mathcal{P}(X)$ and $\langle I^P \rangle_{\mathcal{A}}^T$ and $\langle \mathcal{E}^P \rangle_{\mathcal{A}}^T$ denote the *truncated* $|P|$-point functions.

Now, the joint probability $\langle I^X \rangle_{\mathcal{B}}$ contains contributions coming from the *long contour* passing through *all*, *some* or *none* of the points in X with closed cycles, *disconnected* from the long contour, passing through the remaining points. If the points in X are sufficiently *well separated* then the terms in (23) can be understood as follows: $(-1/2)^{|X'|} \langle \mathcal{E}^{X'} \rangle_{\mathcal{B}}^{\text{con}}$is the probability (up to an unimportant prefactor) of the long contour passing through *all* the points in $X' \subseteq X$ whereas $\langle I^{X \backslash X'} \rangle_{\mathcal{A}}$ is the probability of contours *disconnected* from the long contour passing through the points in $X \backslash X'$. This identification is clear from the truncation properties of $\langle \mathcal{E}^{X'} \rangle_{\mathcal{B}}^{\text{con}}$ and the translational invariance of $\langle I^{X \backslash X'} \rangle_{\mathcal{A}}$ (which is dominated by small bulk-like bubbles passing through the points in $X \backslash X'$). Furthermore, $\langle I^P \rangle_{\mathcal{A}}^T$ in (24) is dominated by the probability of a *single* closed contour passing through *all* the points in P from which one can extract a large-deviations rate functional of Wulff type.

So, on a large enough scale, the joint probabilities can be obtained from the truncated n-point bond-energy correlation functions, $\langle \mathcal{E}^X \rangle^T_{\mathcal{A}}$ and $\langle \mathcal{E}^X \rangle^{\rm con}_{\mathcal{B}}$, which can be evaluated *exactly* using transfer-matrix methods [20]. The results can be framed in terms of path summations as follows [11]. Let $\Gamma(X) = \{[i_1, i_2], [i_2, i_3], \ldots, [i_{n-1}, i_n]\}$ be the path defined as a sequence of line elements (with $[i_j, i_{j+1}]$ connecting the two lattice sites at i_j and i_{j+1}) where $\{i_1, \ldots, i_n\}$ is some *permutation* of X. If $\Gamma_{\rm r}(X)$ is the *rooted* path with i_1 (respectively i_n) connected to the lattice site $(0, 0)$ [respectively $(x, 0)$] then $\langle \mathcal{E}^X \rangle^{\rm con}_{\mathcal{B}}$ can be expressed as a sum over all distinct rooted paths $\Gamma_{\rm r}(X)$ with each term in the sum corresponding to a different way the long contour can pass through the n points in X. Similarly, $\langle \mathcal{E}^X \rangle^T_{\mathcal{A}}$ can be expressed as a sum over all distinct closed circuits $\Gamma_{\rm c}(X) = \{\Gamma(X), [i_n, i_1]\}$.

On a large scale, the path sum for $\langle \mathcal{E}^X \rangle^{\rm con}_{\mathcal{B}}$ is dominated by the *directed path*, i.e., $\Gamma_{\rm r}(X)$ having $0 < x_{i_1} < x_{i_2} < \cdots < x_{i_n} < x$, with all other paths, containing overhangs, being subdominant by a factor of $O(e^{-\ell_{\rm oh}/\xi_{\rm b}})$ where $\ell_{\rm oh}$ is the total excess length of the overhangs in the x direction. To suppress these overhangs one requires that $|x_k - x_j| \gg \xi_{\rm b}$ for all $\{j, k\} \subset X$ and in this limit, with $w - 1$ close to zero when positive, one can show that $(-1/2)^{|X|} \langle \mathcal{E}^X \rangle^{\rm con}_{\mathcal{B}}$ reduces to the product given by (18) with (22).

4.2 SOS Model

Here, the family of finite-dimensional distributions is given by $\mathbf{Q}_x([0, \infty)^{1+x} | Y_{x_1} \in A_1, \ldots, Y_{x_n} \in A_n)$, where $\{x_1, \ldots, x_n\} \subset \{1, \ldots, x-1\}$, which can be *exactly* evaluated using the transfer-integral methods of Ref. [12]. One then applies standard asymptotic methods to the resulting expression for large x with $x_{j+1} - x_j \gg 1$ and we keep $a - 1/\kappa$ small when positve. This leads asymptotically to the joint probability density function given by (18) where $K(\cdot)$ is given by (22).

5 Conclusions and Discussion

Using *exact* methods we have confirmed that a "continuous interface model" describes wetting in the two-dimensional Ising model (and a corresponding lattice SOS model). In the continuum limit, the interfacial path is distributed as a Brownian motion off a *reflecting* barrier containing a Dirac δ perturbation. One needs to be on sufficiently large length scales (with T sufficiently close to T_{w} when $T < T_{\mathrm{w}}$) to get a well defined continuous interface model; i.e., we require that all lengths (including the mean thickness of the wetting layer) be sufficiently larger than the bulk correlation length ξ_{b} for the Ising model and sufficiently larger than 1 (in units of lattice spacing) for the SOS model. All properties of critical wetting (associated with the long contour) in the *asymptotic scaling regime* (such as, e.g., the scaling limit of the complete hierarchy of the n-point correlation functions) are contained within $\mathbf{P}_x^c$.

Our resulting interface "potential" is *not* a function of the type given in Eq. (2) but rather a Dirac δ *distribution* supported on the substrate — indeed, the exact Ising analysis indicates that, on lengths scales of the required size needed to get a well defined interface model, a distribution-valued potential is all one could hope to find. However, on these scales the potential (2) converges in some sense to something resembling a δ distribution [21] although this approach is unlikely to determine the parameter c exactly nor does it explain why the substrate is a reflecting barrier if and only if $T = T_{\mathrm{w}}$. One could still question whether potentials given by (2) should be applied to critical wetting in $d = 2$ bearing in mind that nonlinear functional renormalization group (NFRG) studies [7, 10] starting from such models do just that and the results are then compared to exact Ising solutions. This is used as an important test on the accuracy of the NFRG method which is principally directed to the more elusive case of $d = 3$. We finish with some additional remarks.

(i) For $c < 0$, the wetting layer thickness is given by $\ell = 1/4\tilde{\tau}|c|$ and therefore $2cL_x$ in (10) can be re-written as $-L_x/2\tilde{\tau}\ell$. From this it follows that the measure $\mathbf{P}^c_x$ is *manifestly* invariant under the scale transformation $\ell \mapsto b\ell$, $x \mapsto b^2 x$ and $Y_s \mapsto bY_{b^2 s}$. This means that as ℓ gets arbitrarily large (T arbitrarily close to T_w from below), one can continue to coarse-grain to an arbitrarily large intermediate scale, provided it is much smaller than ℓ, without changing the form of the interface model. This cannot be said of $V(y)$ given by (2) whose range is set by ξ_b.

(ii) The expectation $\lambda := \lim_{x\to\infty} \mathbf{E}\, L_x/x$ provides a measure of the average proportion of the substrate staying close to the interface in the thermodynamic limit. It follows from (11) and (14) that $2\lambda = \partial\tau^\times/\partial c$ from which we have that $\lambda = 2\tilde{\tau}|c|$ for $c < 0$ and $\lambda = 0$ for $c > 0$. Hence, we can see that no matter how close one is to the wetting transition for $T < T_\mathrm{w}$, some proportion of the interface (which gets vanishingly small as $T \uparrow T_\mathrm{w}$) will stay close to the substrate and this *recurrent* property of the interface [22] is not evident from looking at the wetting layer thickness (where $\ell \to \infty$ as $T \uparrow T_\mathrm{w}$) alone. In the mean-field picture, ℓ sits in the minimum of $V(y)$ given by (2) which diverges like $\ln(T_\mathrm{w}^\mathrm{mf} - T)^{-1}$ as $T \uparrow T_\mathrm{w}^\mathrm{mf}$ with no account taken of recurrent events.

(iii) For wetting in the planar Ising model in the presence of a bulk magnetic field, the interface model can be used to make some *exact* scaling-limit predictions. On length scales larger than ξ_b, a positive magnetic field h (in units of $k_\mathrm{B}T$) couples to the total magnetisation difference as given by the area enclosed under the interface. Therefore, defining $\bar{h} = 2m^* h$ (where $m^* > 0$ is the spontaneous magnetisation and h is vanishingly small) the factor $\exp\left(-\bar{h}\int_0^x Y_s\, ds\right)$ is included in the expression for

$\mathbf{P}^c_x$ in (10) and the partition function (11) is similarly modified so that $\mathbf{P}^c_x(\Omega_x) = 1$. ¿From this follows the scaling behaviour

$$\tau^\times(c,h) - \tau = -\Upsilon(2\tilde{\tau}c\xi_h)/2\tilde{\tau}\xi_h^2 \quad \text{with} \quad \xi_h := (4\tilde{\tau}m^* h)^{-1/3} \tag{25}$$

and the scaling function $\Upsilon(z)$ is defined implicitly through $\mathrm{Ai}'(\Upsilon) = z\mathrm{Ai}(\Upsilon)$ where $\mathrm{Ai}(\cdot)$ is the Airy function. This scaling behaviour was found in Ref. [12] for the SOS model but we claim that it also holds in the scaling limit ($T \to T_{\mathrm{w}}^{\pm}$ and $h \downarrow 0$) for the Ising model after putting $c = (1-w)/2\tilde{\tau}$. Also, defining $\lambda = \lambda(c,h)$ as in Remark (ii), we have $2\lambda(c,h) = -\Upsilon'(2\tilde{\tau}c\xi_h)/\xi_h$.

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